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EFFECTIVE INFLOW VELOCITIES INTO A PROPELLER OPERATING IN AN AXISYMMETRIC SHEAR FLOW

by

Theodore R. Goodman

and

Daniel T. Valentine

FINAL REPORT

1 December 1978 — 30 November 1979

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ABSTRACT

An analysis and computer code are presented to calculate the effective wake fraction for a propeller in a nonuniform free wake. The existence of an effective wake has been known for a long time, but the rational theory presented here seems to have eluded all previous investigators. The simple formula developed herein to calculate the effective wake requires knowledge only of the nominal wake and the load distribution on the propeller. Since both the nominal wake and the load distribution are ordinarily given in the design problem, this information suffices to calculate the effective wake for a given design once and for all.

KEYWORDS

Propeller

Ship Wake

TABLE OF CONTENTS

NOMENCLATURE

A	constant in nominal wake distribution; see Eq.(22)
B	constant in nominal wake distribution; see Eq.(22)
C_T	thrust coefficient based on ship speed
J	advance ratio
p	pressure
ΔP	pressure jump on actuator disc
r	radial coordinate
R	propeller radius
$U(r)$	nominal wake velocity
U_e	effective wake velocity
u	axial velocity
u_o	induced axial velocity at actuator disc
U_o	ship speed
V	reciprocal of U
v	radial velocity
x	axial coordinate
Γ	pseudo-stream function; see Eqs.(11),(12),(13)
ρ	fluid density
φ	p/U
ψ_o	stream function at actuator disc; see Eq.(2)

INTRODUCTION

A propeller operating in the wake of a marine vehicle interacts with the wake thereby modifying the flow into the propeller from that of the nominal wake measured in the absence of the propeller. The resulting change in the velocity distribution at the propeller plane is due to the change in pressure induced by the propeller in the wake flow of the vehicle. The radius of many ship propellers is about the same order of magnitude as the thickness of the wake at the propeller location. The propeller suction therefore increases the velocity in the wake in the vicinity of the propeller. This effective inflow velocity distribution ("effective wake") which is experienced by the propeller is a result of the interaction of the propeller and the wake and it is different from the nominal velocity distribution (nominal wake) determined in the absence of the propeller. In order to design a wake-adapted propeller for specified propulsion requirements and a desired radial distribution of loading, it is important to have an estimate of the effective wake.

Huang, et.al.,[1*] have shown experimentally that the conventional technique of adjusting the nominal wake by a constant factor, which is determined from resistance, self-propulsion, and open-water experiments such that the volumetric average effective wake is equal to the thrust identity wake, is not adequate for propellers operating near the stern of axisymmetric bodies. Albeit the conventional procedure provides a reasonable estimate of the average inflow to the propeller, it is unable to determine the actual radial distribution of this inflow. The differences between the effective wake and the nominal wake were found (experimentally) to be greatest near the propeller hub and smallest in the vicinity of the propeller tip. This result is in contrast to the result obtained by the conventional procedure, which usually yields the largest differences between the two wakes in the vicinity of the propeller tip. Therefore, analytical procedures are needed to provide a more accurate description of the interaction between a propeller and its

*Superior numbers in text matter refer to similarly numbered references listed at the end of this report.

inflow. This is especially true because the nature of the effective inflow into the propeller disc is known to play a critical role in predicting powering and cavitation performance.

The investigation reported by Huang, et.al., represents the most comprehensive study in recent years on the steady propeller-hull interaction problem. They presented a prediction method which obviates the conventional technique of determining the effective wake distribution into the propeller operating near the stern of an axisymmetric body. They utilized a moderately-loaded lifting-line model of the propeller to calculate the propeller induced velocities (the model was originally developed for an inflow containing no vorticity; see, for example, Lerbs[2]). The circumferential average velocity distribution is taken to be given two-diameters upstream, where the influence of the propeller is assumed negligible. The flow field from this location up to the propeller disc is calculated numerically by solving a simplified form of the Euler equation of motion for an inviscid fluid containing vorticity. The propeller induced flow field displaces the streamlines from the original position when the propeller was not present. It is the redistribution of the upstream vorticity by the contraction of the streamlines which causes an additional induced velocity field and it is this induced velocity which is the effective wake fraction. In the Huang, et.al. procedure, the effective wake is taken as the inflow into the propeller. Since the effective wake fraction is not known a priori,an iterative scheme was required in which the propeller influence on its inflow was first calculated as if there were no effective wake fraction at the disc, and continued until the prediction of effective wake converged. The comparison with experiments showed good agreement with the lightly- to moderately-loaded propellers considered.

Recently, Goodman[3] developed a linearized axial momentum theory for a lightly-loaded actuator disc operating in an axially directed shear flow. He examined the effect of the vorticity contained in the shear flow experienced by the propeller on the optimum propeller characteristics. He found that the velocities induced by the propeller at its location were substantially altered due to the shear. The results he presented were for an

optimum disc loading only although he presented formulae for arbitrary disc loadings.

The investigation described herein has the purpose of developing a theoretical analysis to compute the effective wake which is assumed to be a consequence of the intensification of the inflow vorticity by the extension of the vortex tubes caused by the propeller induced flow. In other words, a theory for predicting the propeller induced velocity field is developed, which takes into account the vorticity in the onset flow. Goodman's theory was extended to compute the propeller induced velocity field due to a propeller with an arbitrary radial distribution of loading operating in a shear flow; therefore, all the assumptions inherent in his axial-momentum, lightly-loaded, actuator-disc theory apply here as well. The induced flows due to the propeller interaction with its inflow can be separated into two components: one due to the disc itself, and the second due to the rearrangement of the onset flow vorticity. The latter is the effective wake fraction. A formula for the effective wake on the disc is derived for an arbitrary distribution of inflow vorticity which may be evaluated by desk calculation. This formula should prove useful in the preliminary design of propellers behind axisymmetric bodies where a first approximation to the effective wake is required quickly (any errors incurred by the lightly-loaded actuator disc theory assumption are expected to be less than the other uncertainties in preliminary design analysis which are taken into account by applying error margins to the estimated resistance of the body, etc.). However, if the design being considered is heavily loaded, then the prediction by the formula presented herein should be viewed with caution. The designer may want to adjust the effective velocity distribution computed herein as if it were for nominal wake until its volumetric mean matches the thrust identity wake. Then, at least partially, the effect of shear may be accounted for in a quick manner as is often desired in preliminary design analysis.

The results presented show that without shear the formulation, and computed results reduce to the linearized actuator disc theory of Hough and Ordway[4] as expected. An attempt was made to compare the results of the present theory with the measurements presented by Huang,et.al., but a large

discrepancy resulted. There are two possible reasons for this discrepancy: The theory is applied at the propeller disc, whereas the measurements were taken somewhat ahead of the disc. Near the disc the axial velocity is increasing rapidly and so at a small distance ahead of the disc the axial velocities may be considerably smaller than they are on the disc itself. The second reason for the discrepancy is that the measurements (and the theory) of Huang, et.al., were made near a propeller mounted on a body of revolution as in a submarine or torpedo. The body with its incumbent boundary condition thus contributes to the modification of the flow near the propeller. The present theory, on the other hand, is for a propeller in a free wake. To be sure, the wake will have been generated by a body, but that body is several propeller diameters ahead of the propeller and therefore does not play a role in the interaction of the propeller and wake. This situation is more likely to be the case for a ship rather than a submarine or torpedo. Thus, what is required to test the present theory is a set of experiments specifically designed for the purpose.

This research was sponsored by the Naval Sea Systems Command, General Hydromechanics Research Program, and administered by the David W. Taylor Naval Ship Research and Development Center under Contract N00014-79-C-0239, Davidson Laboratory Project 4679/056.

ANALYSIS

The computation of the "effective wake" from a hydrodynamic viewpoint involves determining the modifications to the flow around the stern of a ship into the propeller due to its operation. On an axisymmetric body where the propeller is in the wake, the disturbance of the propeller is felt no more than two-propeller diameters upstream of the propeller plane.[1] The required input data for the determination of the "effective wake" are the nominal wake velocities. These data can either be measured experimentally or computed analytically for a particular hull. A calculation procedure for the flow near the tail region of a body of revolution has recently been presented by Geller.[5] Axisymmetric bodies will be investigated in order to focus on the physical nature of the complex interation between a propeller and a thick wake. Their geometric simplicity offers considerable computational convenience in treating the fundamental aspects of the interaction physics.

In this investigation the nominal wake is assumed to be given. In addition, the nominal wake is usually supplied in the design problem of the propeller. The velocity field induced by the propeller is to be calculated by using the theoretical procedure described next.

Induced Axial Velocities in the Plane of the Propeller

It is shown by Goodman[3] that the pressure jump across the propeller disc in a nonuniform stream U(r) is given by

$$\frac{\Delta P}{2\rho} = Uu_o + \frac{1}{r}\frac{dU}{dr}\psi_o \qquad (1)$$

where

$$\psi_o = -\int_o^r ru_o dr \qquad (2)$$

defines a Stokes stream function, and u_o is the perturbation axial velocity in the plane of the propeller. Equations (1) and (2) give a relationship between the pressure jump ΔP and the induced velocity u_o. Our object will be to invert the relationship so that the induced velocity is expressed

directly in terms of the pressure jump. Solving for ψ_o, we obtain

$$\psi_o \equiv - \int_o^r r u_o dr = \frac{\frac{\Delta P}{2\rho U} - u_o}{\frac{1}{U}\frac{1}{r}\frac{dU}{dr}} \tag{3}$$

Let $\frac{1}{r}\frac{1}{U}\frac{dU}{dr} = f(r)$, $\frac{\Delta P}{2\rho U} = g(r)$, then upon differentiating Eq.(3) with respect to r , there is obtained

$$f \frac{du_o}{dr} - [f' + rf^2]u_o = fg' - gf' \tag{4}$$

This is a first order linear differential equation for u_o, and its solution can be shown to be

$$u_o = \frac{\Delta P}{2\rho U} + \frac{1}{r}\frac{dU}{dr}\int_o^r \frac{\Delta P r dr}{2\rho\ U^2} \tag{5}$$

Thus the total velocity in the plane of the propeller is

$$U + u_o = U + \frac{\Delta P}{2\rho U} + \frac{1}{r}\frac{dU}{dr}\int_o^r \frac{\Delta P r dr}{2\rho\ U^2} \tag{6}$$

The stream velocity U is frequently called the nominal wake, while the second term on the right-hand side is the induced velocity in the absence of shear. The third term is the induced velocity due to the presence of shear, and it is customary in propeller design to lump the first and third terms together and call them an effective wake. Thus,

$$U_e = U + \frac{1}{r}\frac{dU}{dr}\int_o^r \frac{\Delta P r dr}{2\rho\ U^2} \tag{7}$$

Note that in a uniform flow the effective and nominal wakes become identical. From Eq.(7) it is possible to calculate the effective wake directly from the load distribution on the propeller. In the design problem the load distribution is ordinarily given so that the effective wake can be calculated for a given design once and for all. This may be contrasted with the procedure of Huang, et.al.,[1] in which the effective wake is calculated from flow variables which are unknown at the outset so that it becomes necessary to iterate between the effective wake and the wake-adapted propeller to which it is designed. Equation (7) is, in fact, so simple that it is easy to

calculate the effective wake by hand using a trapezoid rule to carry out the integration. The simplicity inherent in Eq.(7) comes about because the equations of motion have been linearized in developing Eq.(1), and also because the nominal wake has been taken to be independent of the axial coordinate. These assumptions make Eq.(1) so simple that it can be inverted. By contrast, Huang, et.al., have not linearized and, consequently, no such inversion is possible in their case. On the other hand, the theory on which the propeller design is based is invariably a linear one so that no great advantage is obtained in retaining nonlinearities in the determination of the effective wake. With regard to the second assumption, viz., that the nominal wake is independent of the axial coordinate, it is certainly true that the nominal wake varies axially according to boundary layer theory and, if the Reynolds number is sufficiently large, the variation will be very small over a distance of one or two propeller diameters. Thus it is sufficiently accurate to take the nominal wake to be independent of the axial coordinate and equal to its value in the plane of the propeller in the absence of the propeller. With these considerations it can be seen that the approximations inherent in the development of Eq.(7) are no more severe than the approximations at the core of the method of Huang, et.al., although they are different.

Of course, the present theory is valid for a propeller in a free wake (as for a ship), whereas the theory of Huang, et.al., is valid for a propeller mounted on a body of revolution (as for a submarine or torpedo). Thus it is not possible to evaluate the present theory using the data of Huang, et.al, and what is needed is an experiment specifically designed to test the present theory.

Induced Axial Velocities Ahead of the Propeller

In order to be able to calculate the induced velocities ahead of the propeller disc, it is necessary to consider the complete linearized field equations. The linearized Euler equations ahead of the disc in the case of axial symmetry are

$$U \frac{\partial u}{\partial x} + v \frac{dU}{dr} = - \frac{1}{\rho} \frac{\partial p}{\partial x} \tag{8}$$

$$U \frac{\partial v}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial r} \tag{9}$$

The equation of continuity is

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial r}(rv) = 0 \tag{10}$$

In order to develop an equation for the axial perturbation velocity in the field, we define a pseudo-stream function Γ such that

$$\frac{p}{\rho} = \frac{U^2}{r}\frac{\partial \Gamma}{\partial r} \tag{11}$$

$$u = -\frac{1}{r}\frac{\partial}{\partial r}(U\Gamma) \tag{12}$$

$$v = \frac{1}{r}\frac{\partial}{\partial x}(U\Gamma) \tag{13}$$

With this definition, Eqs.(8) and (10) are satisfied identically. Expanding Eq.(12), we see that

$$\begin{aligned} u &= -\frac{U}{r}\frac{\partial \Gamma}{\partial r} - \frac{\Gamma}{r}\frac{dU}{dr} \\ &= -\frac{1}{U}\frac{p}{\rho} - \frac{\Gamma}{r}\frac{dU}{dr} \end{aligned} \tag{14}$$

or, from Eq.(11)

$$u = -\frac{1}{U}\frac{p}{\rho} - \frac{1}{r}\frac{dU}{dr}\int_0^r \frac{\frac{p}{\rho}\, rdr}{U^2} \tag{15}$$

Equation (15) may be looked upon as a generalization of Eq.(5) to points ahead of the propeller. Indeed, on the propeller disc $p = -\frac{1}{2}\Delta P$, and, upon substituting into Eq.(15), it can be seen that this equation reduces identically to Eq.(5). From (14) or (15), the effective wake can be seen to take one or the other of the following two forms

$$U_e = U - \frac{1}{r}\frac{dU}{dr}\int_0^r \frac{\frac{p}{\rho}\, rdr}{U^2} \tag{16}$$

$$= U - \frac{1}{r}\frac{dU}{dr}\Gamma \tag{17}$$

The effective wake can therefore be determined ahead of the propeller once p/ρ or Γ are known ahead of the propeller. One possibility is to use pressure measurements ahead of the propeller and to substitute such

measurements into Eq.(16). We will be concerned here, however, with an analytical procedure, and for this purpose will require the solution of the field equation for either Γ or p/ρ. First consider the field equation for Γ.

Upon substituting Eqs.(11) and (13) into Eq.(9), there is obtained

$$\frac{\partial}{\partial x}\left(\frac{U^2}{r}\frac{\partial \Gamma}{\partial x}\right) + \frac{\partial}{\partial r}\left(\frac{U^2}{r}\frac{\partial \Gamma}{\partial r}\right) = 0 \tag{17}$$

In order to obtain the field equation for p, we differentiate Eq.(10) *with respect to* x *and substitute* Eqs.(8) *and* (9). This yields

$$\frac{\partial^2 \; p/\rho}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial p/\rho}{\partial r}\right) + \frac{dU}{dr}\frac{\partial v}{\partial x} - \frac{1}{U}\frac{dU}{dr}\frac{\partial p/\rho}{\partial r} = 0 \tag{18}$$

Then using Eq.(9) again to eliminate $\frac{\partial v}{\partial x}$, the following field equation for p is found

$$\nabla^2 p - 2\,\frac{U'}{U}\frac{\partial p}{\partial r} = 0 \tag{19}$$

This is basically the same equation for the pressure found by von Karman and Tsien.[6] It can be manipulated into another form more suitable for our purposes. Let

$$\begin{aligned} V &= \frac{1}{U} , \\ \varphi &= \frac{p}{U} \end{aligned} \tag{20}$$

It can then be shown that Eq.(19) reduces identically to the following equation for φ:

$$V\,\nabla^2\varphi - \varphi\nabla^2 V = 0 \tag{21}$$

and this equation for φ can be shown to be valid whether or not U and φ also depend on the azimuthal angle as well as the radial coordinate.

We will choose to solve Eq.(21) rather than either Eqs.(17) or (19) because for a particular family of wakes it admits of a relatively simple solution. It can be seen from Eq.(21) that whenever the wake function V obeys the Laplace equation, the pressure function φ will also obey the Laplace equation. In the case of axial symmetry the solution of the Laplace

equation that V must satisfy is of the form

$$\frac{1}{U} \equiv V = A \ln r + B \qquad (22)$$

where A and B are constants. It will be assumed for the purposes of this development that the nominal wake is, indeed, of the form of Eq.(22). In this case, since φ obeys the Laplace equation, we can immediately write down the solution for the pressure in the field from Lamb[7] (Article 102, #2):

$$p = -\frac{U(r)}{2}\int_0^\infty\int_0^\infty J_o(kr)J_o(ks)e^{-k|x|}\frac{\Delta P(s)}{U(s)}\,ksdsdk \qquad (23)$$

where ΔP is the jump in pressure across the disc (= $\Delta P(s)$ for $s < b$, = 0 for $s > b$).

Thus from Eq.(16), the effective wake is

$$U_e = U + \frac{1}{2r}\frac{dU}{dr}\int_0^r dr_1 \int_0^\infty ds \int_0^\infty dk \frac{\frac{\Delta P(s)}{\rho}\, sr_1kJ_o(kr_1)J_o(ks)e^{kx}}{U(s)U(r)} \qquad (24)$$

Now the r_1 integral, viz.,

$$I = \int_0^r \frac{J_o(kr_1)r_1dr_1}{U(r_1)} ,$$

can be evaluated explicitly when U obeys Eq.(22), and the result is (see Appendix A):

$$I = \frac{r\,J_1(kr)}{k\,U(r)} - \frac{A[1 - J_o(kr)]}{k^2} \qquad (25)$$

Thus the effective wake becomes

$$U_e = U - \frac{1}{2}\frac{AU(r)}{r}\int_0^b \frac{\Delta P(s)sI_1(s,r)ds}{\rho U(s)}$$

$$+\frac{1}{2}\frac{A^2U^2(r)}{r}\int_0^b \frac{\Delta P(s)sI_2(s,r)ds}{\rho U(s)} \qquad (26)$$

where

$$I_1 = \int_0^{\infty} J_o(ks)J_1(kr)e^{kx}dk \tag{27}$$

$$I_2 = \int_0^{\infty} \frac{\left(1 - J_o(kr)\right)}{kr} J_o(ks)e^{kx}dk \tag{28}$$

and use has been made of the relationship

$$\frac{dU}{dr} = - \frac{AU^2}{r} \tag{29}$$

which follows from Eq.(22).

It can be shown by using identities involving the Bessel functions that $I_2 = \frac{1}{r}\int_0^r I_1 dr$, and then, passing to the limit $x \to 0$, i.e., on the disc, from Gradshteyn and Ryztik[8] (#6.5123):

$$\left.\begin{aligned} I_1 &= \frac{1}{r} \qquad r > s \quad , \\ &= 0 \qquad r < s \quad , \end{aligned}\right\} \quad x = 0$$

and, consequently,

$$\left.\begin{aligned} I_2 &= \frac{1}{r}\,\ell n\,\frac{r}{s} \qquad r > s \quad , \\ &= 0 \qquad\qquad r < s \quad , \end{aligned}\right\} \quad x = 0$$

I_2 may also be written

$$\left.\begin{aligned} I_2 &= \frac{1}{Ar}\left(\frac{1}{U(r)} - \frac{1}{U(s)}\right) \qquad r > s \quad , \\ &= 0 \qquad\qquad\qquad\qquad\quad r < s \quad , \end{aligned}\right\} \quad x = 0$$

and in this form it is easy to show that Eq.(26) reduces identically to Eq.(7) on the disc as it must.

COMPUTER CODE

A FORTRAN program was developed to compute the velocities given by Eqs.(7) and (26) depending upon the location of the field point of interest. If the field point is on the disc, i.e., at $x=0$ and any r, then Eq.(7) is used and $U=U(r)$ may be any arbitrary distribution of inflow velocity (nominal wake). However, if the field point is off the disc, i.e., $x \neq 0$ and any r, then the inflow velocity distribution must be approximated by Eq.(22). The input radial distribution of the thrust must gradually drop to zero at the hub and tip of the disc, i.e., a disc with constant load may not be considered. This is not a severe restriction because the latter is unrealistic. This restriction is implicit in the assumptions made to develop Eq.(26); see Lamb.[7]

Appendix B presents a listing of the code. The input/output information is described next.

INPUT

The first input card is read in the main program VFSHEAR; see page B1. The constant to be read is an integer called NCASES with format I5. It indicates the number of sets of input cards to follow this card, i.e., it is the number of cases. One case constitutes the complete set of input cards read in the subroutine INPUT; see page B3 in Appendix B.

The set of INPUT cards for each case is clearly presented in the FORTRAN statements on page B3. The appropriate formats are given as well. Therefore, the only additional information required to execute the program is the definition of each piece of input data. The FORTRAN variables representing the input data are defined in Table 1.

TABLE I

INPUT TO VFSHEAR
DEFINITION OF FORTRAN VARIABLES IN THE ORDER THEY ARE READ

FORTRAN VARIABLE	TYPE	DESCRIPTION
IDENT(I)	Alphanumeric	Any phrase used to define the case to be calculated.
RHOF	Real	Fluid density, ρ
DS	Real	Annular strip width (taken to be 0.05 if the hub radius is 0.2R).
AWK	Real	The wake parameter A in Equation (22).
CTHRUST	Real	The thrust coefficient, C_{th}
UNORM	Real	Volumetric average of the input velocities (use 1.0 if field points are off the disc)
NSTRIP	Integer	Number of annular strips (16).
S(I)	Real	Radial coordinate of the center of each strip.
US(I)	Real	Axial inflow velocity at each strip.
DELPS(I)	Real	Pressure jump across each strip.
NOFPS	Integer	Number of field points to be calculated.
N	Integer	Number of Laguerre coefficients in Guass-Laguerre quadruture scheme (50).
XF(J),RF(J)	Real	Coordinates of the field points of interest.
UR(J)	Real	Axial inflow velocity at the field point.

OUTPUT

First, the number of cases is printed followed by a review of the input data for the case under consideration. UR is read as an integer if the point is on the disc. In addition, when computing points on the disc, the field points must be selected such that they are at the center of the strip. The integer value of UR is the number of the radial strip at which the induced velocity is to be computed. UR = 1.0 corresponds to the strip closest to the hub, while UR = 16.0 corresponds to the strip nearest the tip. If the field points are off the disc, then the actual velocity is input for UR.

The first set of computed output is called "velocity at given field points" and is calculated using Eq.(26). If AWK = 0.0, then the values of U do not account for shear and are the potential part of the propeller induced flow.

The second set of output apply only when the field points are taken on the disc. The formula used to calculate the second set of U is Eq.(7).

If the program is executed for an arbitrary distribution of U(r) for field points on the disc, AWK should be set to zero. Then, if the first set of U's are substracted from the second set of U's, the user will obtain the linearized actuator disc theory prediction of the effective wake fraction.

For field points off the disc, the effect wake fraction is determined by running two cases. In the first case AWK is set to its appropriate value for the input US(I). In the second case everything else is kept the same but AWK is set to zero. The differences in the finite values of the output U's is the effective wake fraction.

RESULTS

Since the present theory is for a lightly-loaded actuator disc, for U = constant, we should recover the results calculated and reported by Hough and Ordway.[4] The results calculated with the present code using a loading distribution close to Hough and Ordway's variable distribution were compared with the data presented by Hough and Ordway and were found to be virtually identical. Differences, which are in the third significant figure, are due to the numerical integration scheme and also because the load distribution that was used accounts for a hub while Hough and Ordway's distribution does not.

Next, the nominal wake presented in Figure 16e of Huang, et.al., was used together with the appropriate values of C_{T_S}(=0.371) and J(=1.25) and the load distribution of Hough and Ordway (modified to account for a hub) to calculate the total axial velocity using Equation (5). It is to be noted that the nominal wake was measured one-quarter of a radius ahead of the propeller disc, but for the purposes of this calculation it was assumed that the nominal wake remained the same at the disc itself. No attempt was made to use the off-disc computer program for this comparison because the wake measured and presented by Huang, et.al. does not fit Eq.(22). Instead, the wake appears to fit the "law of the wall," viz., $U = a\,\ell n\, r + b$, and this fact was used to extrapolate the nominal wake data presented to the hub of the propeller. The results are presented in Figure 1. A set of experiments specifically designed to test the present theory is clearly required. Since it is impossible to measure the axial velocity directly at the propeller disc, it would be necessary to measure it a small distance ahead and an equal distance behind the disc. Then, since the axial velocity is known to be monotonic and continuous through the disc, its value at the disc can be inferred by averaging the fore and aft measurements. Moreover, in order to simulate the wake of a body, a wake screen could be used and the measurements taken in a water tunnel using a Lazer-Doppler Anemometer.

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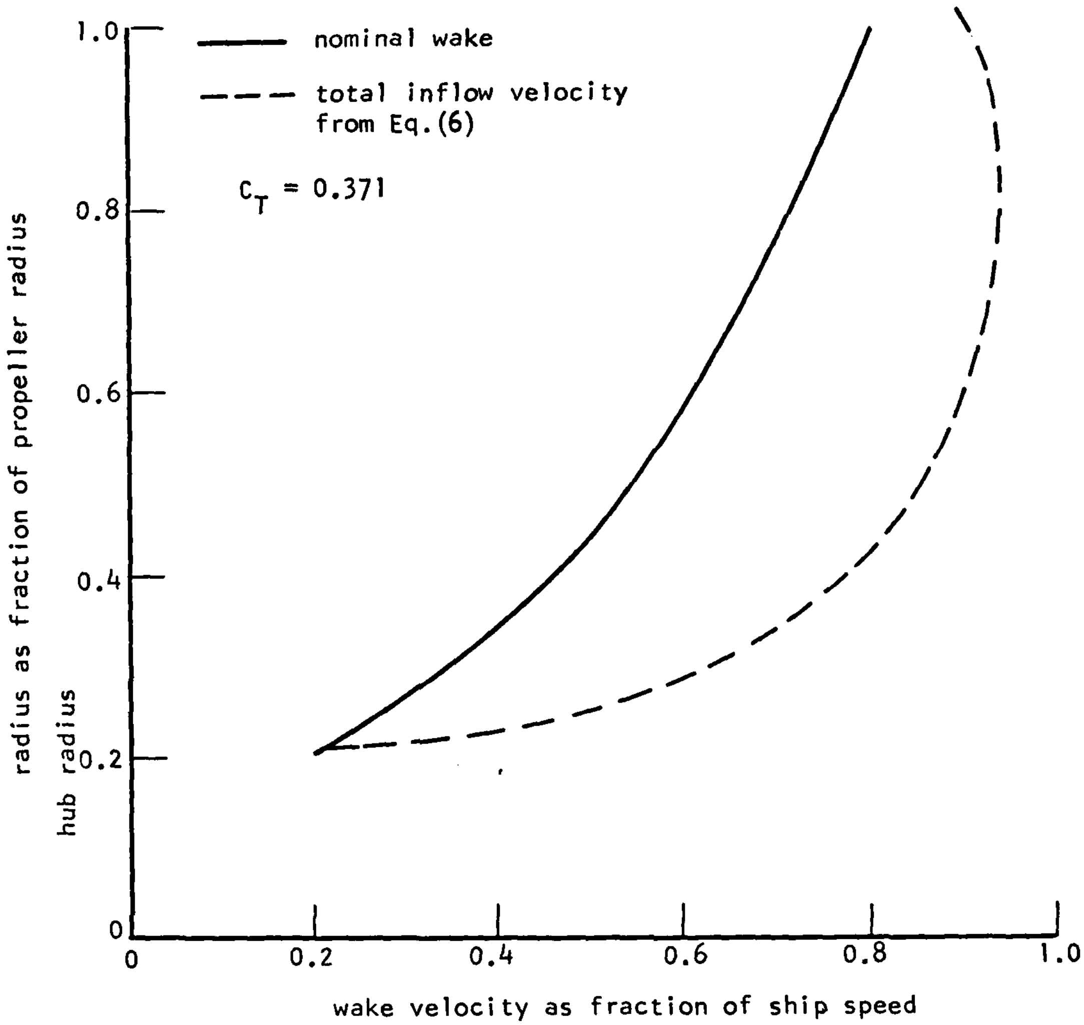

FIGURE 1. NOMINAL WAKE AND TOTAL INFLOW VELOCITY FROM FIGURE 16e OF REFERENCE [1] AND EQUATION (6)

APPENDIX A

EVALUATION OF A CERTAIN INTEGRAL

Consider the integral

$$I = \int_0^r \frac{J_o(kr_1)\,r_1 dr_1}{U(r_1)}$$

where

$$\frac{1}{U} = A \ln r_1 + B$$

$$= A \ln kr_1 + B'$$

where

$$B' = B - A \ln k \quad .$$

Let

$$kr_1 = z_1$$

$$kr = z$$

Then

$$I = \frac{1}{k^2} \int_0^z z_1 dz_1 [A \ln z_1 + B'] J_o(z_1)$$

but

$$z_1 J_o(z_1) = \frac{d}{dz_1}\left(z_1 J_1(z_1)\right) \quad .$$

Upon substituting for J_o, the term proportional to B can be integrated directly, while the term proportional to A can be integrated by parts. This yields

$$I = \frac{1}{k^2}\left\{AzJ_1(z) \ln z - A \int_0^z J_1(z_1) dz_1 + B' zJ_1(z)\right\}$$

But since $J_1 = -J_o'$, the integral can be evaluated. Then, upon substituting for B' in terms of B and simplifying, the result, Equation (25), follows.

APPENDIX B. FORTRAN LISTING OF PROGRAM VFSHEAR

```
      PROGRAM VFSHEAR(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE7,
     * TAPE1,TAPE3)
C
C     PROGRAM TO COMPUTE VELOCITY FIELD UPSTREAM OF A PROPELLER
C     OPERATING IN A SHEAR FLOW DEFINE BY  1/U = AWK*LOGE(R) + BWK
C
      COMMON/INPUT/RHOF,S(32),DS,XF(100),RF(100),DELPS(32)
      COMMON/NOMWAKE/US(32),NOFPS,UR(100),U(100),AWK
      COMMON/INTGRND/FI0,FI1,FI2,XI0,XI1,XI2
      DIMENSION FI0(960),FI1(960),FI2(960)
      DIMENSION XI0(32),XI1(32),XI2(32)
      COMMON/LAGZRS/ Y(15),N
      COMMON/DUDR/ URX(100)
      COMMON/THRUST/ CTHRUST,UNORM
      COMMON/STRIP/ NSTRIP
      COMMON/LAG/XL,AL,NN
      DIMENSION Q(125),Q1(125)
      DIMENSION XL(50),AL(50),B(50),C(50)
C
C     READ INPUT DATA
C
      READ(5,2000) NCASES
 2000 FORMAT(I5)
      WRITE(6,2002) NCASES
 2002 FORMAT(///*   NUMBER OF CASES =*I5/)
      DO 2001 INCI=1,NCASES
      CALL INPUT
C     FOR EACH FIELD POINT COMPUTE VELOCITY
C**
      IF(US(1).EQ.0.0) BWK=US(2)
      IF(US(1).NE.0.0) GO TO 95
      DO 96 IXN=1,NSTRIP
   96 US(IXN)=1.0/(AWK*ALOG(S(IXN))+BWK)
   95 CONTINUE
C
C     CALCULATION OF ZEROS AND COEFFICIENTS IF N.NE.15.OR.N.NE.10.
C
      EPS = 0.0000000001
      EPS = EPS**2
      IF(N.EQ.15.OR.N.EQ.10) GO TO 546
      NN = N
      DO 547 I = 1,NN
      XI = FLOAT(I)
       B(I) = 2.*XI-1.
  547 C(I) = (XI-1.)**2
      CALL LAGUER(NN,XL,AL,0.0,B,C,EPS,CSX,CSA,TSX,TSA)
  546 CONTINUE
      DO 1001 IFP=1,NOFPS
      IF(XF(IFP).EQ.0.0) GO TO 1019
C
C     CALCULATE INGREDIENTS TO VELOCITY INTEGRATIONS
C
      IFPX = IFP
      CALL INTGRD(IFPX)
C
C     PERFORM GAUSSIAN-LAGUERRE QUADRATURE
C
      CALL GAUSSLG
C
C     AT THIS POINT WE HAVE FOR EACH STRIP I0,I1,AND I2
      IF(XF(IFP).GT.0.2) GO TO 2090
      DO 10001 IX=1,NSTRIP
```

```
      ARGQ=(XF(IFP)**2+RF(IFP)**2+S(IX)**2)/(2.*S(IX)*RF(IFP))
      FACQ=XF(IFP)/((RF(IFP)*S(IX))**1.5)
      CALL LEG2HO(ARGQ,2,Q,Q1,2)
      XIO(IX)=Q1(1)*FACQ/3.1415927
      XIO(IX)=-XIO(IX)
10001  CONTINUE
 2090 CONTINUE
C
C     CALCULATE VELOCITY AT PARTICULAR FIELD POINT
C
C     SPANWISE INTEGRATION
C
            USUM0=0.0
      USUM1=0.0
          USUM2 = 0.0
      DO 1002 JS=1,NSTRIP
      FAC = DS*DELPS(JS)/(RHOF*US(JS)*2.)
      UINTG = S(JS)*FAC*XIO(JS)
      USUMO = USUMO + UINTG
      UINTG1 = S(JS)*FAC*XI1(JS)
      USUM1 = USUM1 + UINTG1
      UINTG2 = S(JS)*FAC*XI2(JS)
 1002 USUM2 = USUM2 + UINTG2
      U(IFP) = USUMO-AWK*UR(IFP)*USUM1/RF(IFP)+AWK*AWK*UR(IFP)*UR(IFP)*
     1 USUM2/RF(IFP)
      GO TO 1060
 1019 CONTINUE
      IF(RF(IFP).LE.1.0) GO TO 1018
      GO TO 1016
 1018 CONTINUE
C**    S = R SOMEWHERE----RESTRICTION
      ISI = UR(IFP)
      UR(IFP)=1.0
      UXO=0.5*DELPS(ISI)/(RHOF*US(ISI))
      UX1=0.0
      DO 1014 IXI=1,ISI
      UX1 = UX1 + DELPS(IXI)*S(IXI)/(US(IXI)**2)
 1014 CONTINUE
      IF(ISI.EQ.1.OR.ISI.EQ.16) GO TO 93
      URXP = (US(ISI+1)-US(ISI-1))/(2.*DS)
      URX(IFP) = UXO +      DS*URXP*UX1/(RF(IFP)*RHOF)
   93 CONTINUE
      UX1 = UX1*AWK*UR(IFP)**2*0.5/(RF(IFP)**2*RHOF)
      UX1 = UX1*DS*(US(ISI)**2)
      UR(IFP) = US(ISI)
      U(IFP) = UXO-UX1
      GO TO 1060
 1016 U(IFP) = 0.0
 1060 CONTINUE
C**
C**   SAMPLE CASE THRUST COEFFICIENT
C**
      U(IFP) = U(IFP)/CTHRUST
      URX(IFP) = URX(IFP)/CTHRUST
C**
 1001 CONTINUE
C
C     PRINT RESULTS
C
      CALL OUTPUT
 2001 CONTINUE
      STOP
```

```
      END
      SUBROUTINE INPUT
      COMMON/INPUT/RHOF,S(32),DS,XF(100),RF(100),DELPS(32)
      COMMON/NCMWAKE/US(32),NOFPS,UR(100),U(100),AWK
      COMMON/LAGZRS/ X(15),N
      COMMON/STRIP/ NSTRIP
      COMMON/THRUST/ CTHRUST,UNORM
C
C     READ INPUT -- RHOF=FLUID DENSITY,S(I)= I STRIPS, DS=WIDTH OF
C     STRIPS, (XF,RF)=COORDS. OF FIR
C     STRIPS, (XF,RF)=COORDS. OF FIELD POINTS, WHERE XF.LT.0.0,  US(I)=
C     INFLOW AT EACH STRIP
C
C     INPUT THE MOD OR ABSOLUTE VALUE OF XF
C
      DIMENSION IDENT(7)
      READ (5,100) (IDENT(I),I=1,7)
      WRITE(6,101)
      WRITE(6,212)
      WRITE(6,100) (IDENT(I),I=1,7)
      READ(5,102) RHOF,DS,AWK,CTHRUST,UNORM
      IF(CTHRUST.EQ.0.0) CTHRUST=1.0
      IF(UNORM.EQ.0.0) UNORM=1.0
      WRITE(6,213)
      WRITE(6,110)
      WRITE(6,111) RHOF,DS,AWK
      READ(5,103) NSTRIP
      READ(5,102) (S(I),I=1,NSTRIP)
      WRITE(6,112)
      WRITE(6,102) (S(I),I=1,NSTRIP)
      READ(5,102) (US(I),I=1,NSTRIP)
      WRITE(6,113)
      DO 1001 IXI=1,NSTRIP
 1001 US(IXI) = US(IXI)/UNORM
      WRITE(6,102) (US(I),I=1,NSTRIP)
      READ(5,102) (DELPS(I),I=1,NSTRIP)
      WRITE(6,114)
      WRITE(6,102) (DELPS(I),I=1,NSTRIP)
      READ(5,103) NOFPS,N
      IF(NOFPS.GT.100) WRITE(6,104) NOFPS
      IF(NOFPS.GT.100) STOP
      WRITE(6,115) NOFPS
      WRITE(6,116)
      DO 105 I=1,NOFPS
  105 READ(5,102) XF(I),RF(I),UR(T)
      DO 117 I=1,NOFPS
       XFA = -XF(I)
  117 WRITE(6,118) XFA,PF(I),UP(I)
C
C
C     FORMAT STATEMENTS FOR INPUT/OUTPUT
C
  101 FORMAT(1H1)
  100 FORMAT(7A10)
  212 FORMAT(//)
  213 FORMAT(///)
  103 FORMAT(8I5)
  110 FORMAT(/10X,* INPUT CARD NO. 2*/)
  111 FORMAT(5X,*  RHOF=*F10.5,*        DS=*F10.5,*        AWK=*F10.5/)
  112 FORMAT(/10X,*   INPUT CARD NO. 3,PROPELLER STRIPS-MIDPOINTS*/)
  102 FORMAT(8F10.5)
  113 FORMAT(/10X,*  INPUT CARD NO. 4,INFLOW AT EACH STRIP*/)
```

```
  114 FORMAT(/10X,*  INPUT CARD NO. 5, STEADY THRUST LOADING AT EACH STR
     1IP*/)
  115 FORMAT(//10X,*    NO. OF FIELD POINTS =*I3)
  116 FORMAT(/10X,*  FIELD POINT COORDINATES*/)
  118 FORMAT(10X,*   XF =*F10.5,8X,*    RF=*F10.5,*     UR=*F10.5)
  104 FORMAT(/5X,*    NOFPS=*I5,*   EXCEEDS DIMENSIONS OF 100*/)
C     UR IS THE VELOCITY OF STREAM AT RF
      RETURN
      END
      SUBROUTINE INTGRD(IFP)
      COMMON/INTGRND/FI0,FI1,FI2,XI0,XI1,XI2
      DIMENSION FI0(960),FI1(960),FI2(960)
      DIMENSION XI0(32),XI1(32),XI2(32)
C
C
C     CALCULATION OF INTEGRANDS FOR I1,I2,AND I0
C
      COMMON/BESFUNC/XJ0S,XJ0R,XJ1R
      DIMENSION XJ0S(960),XJ0R(50),XJ1R(50)
      COMMON/INPUT/RHOF,S(32),DS,XF(100),RF(100),DELPS(32)
      COMMON/NOMWAKE/US(32),NOFPS,UR(100),U(100),AWK
C**   COMMON/LAGZRS/ Y(15),N
      COMMON/LAGZRS/ YY(15),NNN
      COMMON/LAG/XL(50),AL(50),NN
      COMMON/STRIP/ NSTRIP
      DIMENSION XJ(10),XJ1(10)
      DIMENSION Y(50)
C**   IF(N.EQ.0) N=15
C
C     GET ZEROS OF LAGUERRE POLYNOMIALS
C
C**   CALL LGZEROS(N)
      N=NN
      DO 300 I=1,N
  300 Y(I)=XL(I)
      IF(N.EQ.10.OR.N.EQ.15) CALL LGZEROS(N)
C
C     CALCULATE BESSEL FUNCTIONS J0 AND J1
C
      DO 100 I=1,N
      DO 101 J=1,NSTRIP
      IJ = NSTRIP*(I-1)+J
      ARG0=Y(I)*S(J)/XF(IFP)
      CALL BESL(1,XJ,0,10,0,ARG0,1)
      XJ0S(IJ)=XJ(1)
  101 CONTINUE
      ARG1=Y(I)*RF(IFP)/XF(IFP)
      CALL BESL(1,XJ1,0,10,0,ARG1,2)
      XJ0R(I)=XJ1(1)
      XJ1R(I)=XJ1(2)
  100 CONTINUE
      DO 200 J=1,NSTRIP
      DO 201 I=1,N
      IJ = NSTRIP*(I-1)+J
C
C     CALCULATE FI0
C
      FI0(IJ)=Y(I)*XJ0S(IJ)*XJ0R(I)/(XF(IFP)**2)
C
C     CALCULATE FI1 AND FI2
C
      FI1(IJ)=XJ0S(IJ)*XJ1R(I)/XF(IFP)
```

```
      FI2(IJ) = (1.-XJOR(I))*XJOS(IJ)/(Y(I)*RF(IFP))
  201 CONTINUE
  200 CONTINUE
      RETURN
      END
      SUBROUTINE LGZEROS(N)
      COMMON/LAGZRS/ X(15),N1
C
C     ZEROS OF THE LAGUERRE POLYNOMIALS
C
C     SELECT N = 10,  OR N = 15 ONLY
C
      DIMENSION X10(10),X15(15)
      DATA((X10(I),I=1,10)=.137793470540,.729454549503,
     11.80834290174,3.40143369778,55,5.5552496140064,
     28.330152746764,11.8437858379,16.279257831378,
     3 21.996585811981,29.920697012274)
      DATA ((X15(I),I=1,15)=0.093307812017,0.492691740302,
     1 1.215595412071,2.269949526204,3.667622721751,
     2 5.425336627414,7.565916226613,10.120228568019
     3,13.130282482176,16.654407708330,20.776478899449,
     4 25.623894226729,31.407519169754,38.530683306486,
     5 48.026085572686)
C
      N1 = N
      DO 100 I=1,15
  100 X(I) = 0.0
      IF(N.EQ.15) GO TO 102
C     IF(N.EQ.10) CONTINUE
      DO 101 I=1,10
  101 X(I)=X10(I)
      GO TO 104
  102 CONTINUE
      DO 103 I=1,15
  103 X(I) = X15(I)
  104 CONTINUE
      RETURN
      END
      SUBROUTINE GAUSSLG
      COMMON/INTGRND/FIO,FI1,FI2,XIO,XI1,XI2
      DIMENSION FIO(960),FI1(960),FI2(960)
      DIMENSION XIO(32),XI1(32),XI2(32)
      DIMENSION A(50),A10(10),A15(15)
      COMMON/LAG/XL(50),AL(50),NN
      COMMON/STRIP/ NSTRIP
C
C     GAUSS-LAGUERRE QUADRATURE
C
      COMMON/LAGZRS/ X(15),N
C
      DATA((A10(I),I=1,10)=.308441115765,.401119929155,
     1 .218068287612,0.062087456098 7,.009501516975 18,
     2 .000753008388588,.0000282592334 96,.0000004249314,
     3 .000000001839565,.00000000000991183)
      DATA ((A15(I),I=1,15)=.21823488594,.342210177923,
     1 .263027577942,.126425818106,.040206864921,.00856387780361,
     2 .00121243614721,.000111674392344,.0000064599267 6202,
     3 .0000002226316 9071,.0000000042274303 8498,
     4 .000000000039218973,.000000000000145652,
     5 .000000000000000148303,.000000000000000000016006)
      IF(N.EQ.55) GO TO 500
      N=NN
```

```
      IF(N.NE.10.OR.N.NE.15) GO TO 106
C     IF(N.EQ.10) CONTINUE
      IF(N.EQ.15) GO TO 101
      DO 100 I=1,10
  100 A(I) = A10(I)
      GO TO 103
  101 DO 102 I=1,15
  102 A(I) = A15(I)
      GO TO 103
  106 DO 108 I=1,N
  108 A(I)=AL(I)
  103 CONTINUE
C
C     XI0,XI1,XI2 INTEGRALS
C
      DO 104 J=1,NSTRIP
      SUM0=0.0
       SUM1=0.0
         SUM2=0.0
      DO 105 I=1,N
      IJ=NSTRIP*(I-1)+J
      SUM0=SUM0+FI0(IJ)*A(I)
      SUM1=SUM1+FI1(IJ)*A(I)
      SUM2=SUM2+FI2(IJ)*A(I)
  105 CONTINUE
      XI0(J) = SUM0
C**    WRITE(6,220) XI0(J)
  220 FORMAT(//*      XI0=*E14.6//)
      XI1(J) = SUM1
      XI2(J) = SUM2
  104 CONTINUE
      GO TO 505
  500 CONTINUE
      N = 15
      SUM0=0.0
      DO 501 I=1,15
      SUM0 = SUM0 + FI0(I)*A15(I)
C**   WRITE(6,5050) SUM0,FI0(I),A15(I)
 5050 FORMAT(2X,*  SUM0=*E14.6,*   FI0=*E14.6,*     A15(I)=*E14.6)
  501 CONTINUE
      XI0(1)=SUM0
      WRITE(6,220) XI0(1)
      RETURN
  505 CONTINUE
      RETURN
      END
      SUBROUTINE BESL(M,J,Y,NDIMJ,NDIMY,X,NMAX)
C
C     CALCULATION OF BESSEL FUNCTIONS
C     M=1, FOR J
C     M=2 ,FOR Y
C     M=3 ,FOR J AND Y
C     NDIMJ-DIMENSION OF J(AT LEAST 2)
C     NDIMY-DIMENSION OF Y
C     X-ARGUMENT
C     NMAX=MAXIMUM ORDER  PLUS ONE
C
      REAL J
      DIMENSION J(NDIMJ),Y(NDIMY),CJ0L3(7),CJ1L3(7),CY0L3(7),CY1L3(7),
     1CF0(7),CTH0(7),CF1(7),CTH1(7)
      DATA ((CJ0L3(I),I=1,7)= 1.0,-2.2499997, 1.2656208,-.31638660,
     1.04444790,-.00394440, .00021000),((CJ1L3(I),I=1,7)=.50      ,
```

```
     2-.56249985, .21093573,-.03954289, .00443319,-.00031761, .00001109)
     3, ((CYOL3(I),I=1,7)= .36746691, .60559366,-.74350384, .25300117,
     4-.04261214, .00427916,-.00024846), ((CY1L3(I),I=1,7)=-.63661980,
     5 .22120910, 2.1682709,-1.3164827, .31239510,-.04009760, .0027873)
      DATA ((CFO(I),I=1,7)= .79788456,-.00000077,-.00552740,-.00009512,
     1 .00137237,-.00072805, .00014476), ((CTHO(I),I=1,7)=-.78539816,
     2-.04166397,-.00003954, .00262513,-.00054125,-.00029333,.00013558),
     3((CF1(I),I=1,7)= .79788456, .00000156, .01659667, .00017105,
     4-.00249511, .00113653,-.00020033 ), ((CTH1(I),I=1,7)=-2.35619449,
     5 .12499612, .00005650,-.00637879, .00074348, .00079824,
     6-.00029166)
      IF(X.EQ.0.) GO TO 9
      PI= 3.1415926
      X1=ABS(X)
      S=X/X1
      T= X*X/9.0
      IF(M.NE.1) FLPI=2.*ALOG(.5*X1)/PI
      IF(X1.GT.3) GO TO 1
      J(1)= POLY(6,CJOL3,T,7)
      IF (M .NE. 1) Y(1)=POLY(6,CYOL3,T,7)+J(1)*FLPI
      IF (NMAX .EQ. 1) RETURN
      IF (M .EQ. 1 .AND. NMAX .GT. 2) GO TO 5
      J(2)=X*POLY(6,CJ1L3,T,7)
      IF (M .EQ. 1) RETURN
      Y(2)= FLPI*J(2)+POLY(6,CY1L3,T,7)/X
      IF (NMAX .EQ. 2) RETURN
      IF(M .EQ. 2) GO TO 6
    5 MM=NMAX+5
      FM=MM
      J(MM+2)=1.0
      J(MM+1)= SQRT(FM*(FM+1.))*((FM+1.)/FM)**(MM+1)*2./(X*2.71828)
      A=J(1)
      MM=MM+2
      CALL BESREC(1,J,NDIMJ,X,MM)
      FAC= A/J(1)
      DO 7 I=1,NMAX
    7 J(I)=FAC*J(I)
      IF (M .EQ. 1) RETURN
    6 CALL BESREC(2,Y,NDIMY,X,NMAX)
      RETURN
    1 T=3./X1
      SQX=1./SQRT(X1)
      F=POLY(6,CFO,T,7)
      TH=POLY(6,CTHO,T,7)+X1
      IF (M .NE. 2) J(1)=F*COS(TH)*SQX
      IF ( M .NE. 1) Y(1)=F*SIN(TH)*SQX
      IF (NMAX .EQ. 1) RETURN
      FNMAX=NMAX-1
      IF (M .EQ. 1 .AND. NMAX .GT. 2 .AND. FNMAX .GT.X1)GO TO 5
      F= POLY(6,CF1,T,7)
      TH= POLY(6,CTH1,T,7)+X1
      IF(M .NE. 2 .AND. FNMAX .LE. X1)J(2)=F*COS(TH)*SQX*S
      IF (M .NE.1) Y(2)=F*SIN(TH)*SQX
      IF (NMAX .EQ. 2) RETURN
      IF (M .EQ. 2) GO TO 6
      IF (FNMAX .GT. X1)GO TO 5
      CALL BESREC(2,J,NDIMJ,X,NMAX)
      IF (M .EQ. 3) GO TO 6
      RETURN
    9 DO 10 I=1,NMAX
   10 J(I)=0.
      J(1)=1.
```

```
      RETURN
      END
      SUBROUTINE BESREC(M,W,NDIMW,X,NMAX)
C     RECURRENCE  FORMULAS FOR BESSEL FUNCTIONS
C     M=1, FOR BACKWARD J AND Y
C     M=2, FOR  FORWARD J AND Y
C     M=3, FOR BACKWARD I
C     M=4, FOR  FORWARD K
C     W(I)= INDICATED BESSEL FUNCTION OF ORDER I-1 AND ARGUMENT X
C     NMAX= LARGEST ORDER  PLUS ONE
C
      DIMENSION W(NDIMW)
      F1(FN,U,V)= 2.*FN*U/X-V
      F2(FN,U,V)= 2.*FN*U/X+V
      IF (M .EQ. 1 .OR. M .EQ. 3) GO TO 1
      IF (M .EQ. 4) GO TO 6
      DO 3 I=3,NMAX
      FI= I-2
    3 W(I)=   F1(FI,W(I-1),W(I-2))
      RETURN
    1 IF (M .EQ. 3) GO TO 4
      DO 2 I=3,NMAX
      J= NMAX-I+1
      FJ=J
    2 W(J)= F1(FJ,W(J+1),W(J+2))
      RETURN
    4 DO 5 I=3,NMAX
      J=NMAX-I+1
      FJ=J
    5 W(J)= F2(FJ,W(J+1),W(J+2))
      RETURN
    6 DO 7 I=3,NMAX
      FI=I-2
    7 W(I)=F2(FI,W(I-1),W(I-2))
      RETURN
      END
      SUBROUTINE MODBES(M,I,K,NDIMI,NDIMK,X,NMAX)
C     COMPUTATION OF MODIFIED BESSEL FUNCTIONS
C     M=1, FOR I
C     M=2, FOR K
C     M=3, FOR I AND K
C     X=ARGUMENT
C     NMAX=MAXIMUM ORDER PLUS ONE
C
      REAL I,K
      DIMENSION I(NDIMI), K(NDIMK), CIOL (7), CI1L (7), CKOL (7),
     1CK1L (7), CIOG (9), CI1G (9), CKOG (7), CK1G (7)
      DATA ((CIOL(I),I=1,7)=1.,3.5156229,3.0899424,1.2067492,.2659732,
     1  .0360768,.0045813),((CIOG(I),I=1,9)=.39894228,.01328592,
     2  .00225319,-.00157565,.00916281,-.02057706,.02635537,-.01647633,
     3  .00392377),((CI1L(I),I=1,7)=.5,.87890594,.51498869,.15084934,
     4  .02658733,.00301532,.00032411),((CI1G(I),I=1,9)=.39894228,
     5  -.03988024,-.00362018,.00163801,-.01031555,.02282967,-.02895312,
     6  .01787654,-.00420059)
      DATA ((CKOL(I),I=1,7)=-.57721566,.42278420,.23069756,.03488590,
     1  .00262698,.00010750,.0000074),((CKOG(I),I=1,7)=1.25331414,
     2  -.07832358,.02189568,-.01062446,.00587872,-.0025154,.00053208),
     3  ((CK1L(I),I=1,7)= 1.,.15443144,-.67278579,-.18156897,-.01919402,
     4  -.00110404,-.00004686),((CK1G(1),I=1,7)=1.25331414,.23498619,
     5  -.0365562,.01504268,-.00780353,.00325614,-.00068245)
      GCOEF(FN,X)= SQRT(FN*FN+X*X)+FN*ALOG(X/(FN+SQRT(FN*FN+X*X)))
      IF (X .LE. 0. .AND. M .NE. 1) GO TO 200
```

```
      IF (X .EQ. 0.) GO TO 99
      X1= ABS(X)
      T2= .25*X*X
      TR=2./X
      T1=X/3.75
      T1S= X*X/14.0625
      T1R= 3.75/X1
      XSR= 1./SQRT(X1)
      DEC= EXP(X1)
      DECK= EXP( -X1)
      FLN= ALOG(.5*X1)
      S= X1/X
      IF (M .EQ. 2)GO TO 100
C
C     MODIFIED BESSEL (FIRST KIND)
C
      IF (X1.GT. 3.75) GO TO 10
      I(1)= POLY(6,CIOL,T1S,7)
      IF (M .EQ. 1 .AND. NMAX .EQ. 1) RETURN
      IF (M .EQ. 3 .AND. NMAX .EQ. 1) GO TO 100
      IF (NMAX .NE. 2) GO TO 1
      I(2)= POLY(6,CI1L,T1S,7)*X
      IF (M .NE. 1) GO TO 100
      RETURN
    1 A=I(1)
      MM= NMAX+5
      FM= MM
      I(MM+1)= 1./SQRT(6.28318  *FM)/((1.+X*X/(FM*FM))**0.25)*
     1EXP(GCOEF(FM,X1))*S
      FM=FM+1
      I(MM+2)= 1./SQRT(6.28318  *FM)/((1.+X*X/(FM*FM))**0.25)*
     1EXP(GCOEF(FM,X1))
      MM= MM+2
      CALL BESREC(3,I,NDIMI,X,MM)
      FAC= A/I(1)
      DO 2 L=1,NMAX
    2 I(L)= FAC*I(L)
      IF (M .EQ. 3) GO TO 100
      RETURN
   10 I(1)= POLY(8,CIOG,T1R,9)*XSR*DEC
      IF (M .EQ. 1 .AND. NMAX .EQ. 1) RETURN
      IF (M .EQ. 3 .AND. NMAX .EQ. 1) GO TO 100
      IF (NMAX .NE. 2) GO TO 1
      I(2)= POLY(8,CI1G,T1R,9)*XSR*DEC*S
      IF (M .NE. 1) GO TO 100
      RETURN
C
C     MODIFIED BESSEL (SECOND KIND)
C
  100 IF (X .GT. 2.) GO TO 110
      IF (M .EQ. 2) I(1)= POLY(6,CIOL,T1S,7)
      K(1)= -FLN*I(1)+POLY(6,CKOL,T2,7)
      IF (NMAX .EQ. 1) RETURN
      IF (M .EQ. 2) I(2)= POLY (6,CI1L,T1S,7)*X
      K(2)= (X*FLN*I(2)+POLY(6,CK1L,T2,7))/X
      IF (NMAX .NE. 2) GO TO 101
      RETURN
  101 CALL BESREC(4,K,NDIMK,X,NMAX)
      RETURN
  110 K(1)= POLY(6,CKOG,TR,7)*XSR*DECK
      IF (NMAX .EQ. 1) RETURN
      K(2)= POLY(6,CK1G,TR,7)*XSR*DECK
```

```
      IF (NMAX .NE. 2) GO TO 101
      RETURN
   99 DO 98 J=1,NMAX
   98 I(J)=0.
      I(1)=1.
      RETURN
  200 PRINT  97,X
   97 FORMAT (10X26HK NOT DEFINED FOR X .LE. 0,2X2HX=E16.8)
      CALL EXIT
      END
      FUNCTION POLY(N,A,X,NDIMA)
      DIMENSION A(NDIMA)
      POLY=A(N+1)
      DO 1 I=1,N
      J=N+1-I
    1 POLY=A(J)+X*POLY
      RETURN
      END
      SUBROUTINE OUTPUT
C
C     WRITE OUT THE COMPUTED VELOCITIES AT THE GIVEN FIELD POINTS
C
      COMMON/INPUT/RHOF,S(32),DS,XF(100),RF(100),DELPS(32)
      COMMON/NOMWAKE/US(32),NOFPS,UR(100),U(100),AWK
      COMMON/DUDR/ URX(100)
      WRITE(6,100)
  100 FORMAT(1H1)
      WRITE(6,101)
  101 FORMAT(/10X,*   VELOCITY AT GIVEN FIELD POINTS*/)
      WRITE(6,102)
  102 FORMAT(/5X,* ABS(XF)        RF            UR                 U */)
      WRITE(6,105) (XF(I),RF(I),UR(I),U(I),I=1,NOFPS)
  105 FORMAT(5X,F10.5,2X,F10.5,2XF10.5,5X,E14.6)
      WRITE(6,100)
      WRITE(6,110)
      WRITE(6,102)
      WRITE(6,105) (XF(I),RF(I),UR(I),URX(I),I=1,NOFPS)
  110 FORMAT(/10X,*   VELOCITY FIELD ON DISK USING EXACT DUDR*/)
      RETURN
      END
      SUBROUTINE LAGUER(NN,X,A,ALF,B,C,EPS,CSX,CSA,TSX,TSA)
      DIMENSION X(50),A(50),B(50),C(50)
      FN = NN
      CSX = 0.0
      CSA = 0.0
      CC = GAMMA(ALF+1.)
      TSX = FN*(FN+ALF)
      TSA=CC
      DO 1 J=2,NN
    1 CC = CC*C(J)
      DO 7 I=1,NN
      IF(I-1) 6,2,3
    2 XT = (1.+ALF)*(3.+.92*ALF)/(1.+2.4*FN+1.8*ALF)
      GO TO 6
    3 IF(I-2)6,4,5
    4 XT = XT+(15.+6.25*ALF)/(1.+.9*ALF+2.5*FN)
      GO TO 6
    5 FI = I-2
      R1=(1.+2.55*FI)/(1.9*FI)
      R2=1.26*FI*ALF/(1.+3.5*FI)
      RATIO=(R1+R2)/(1.+.3*ALF)
      XT = XT + RATIO*(XT-X(I-2))
```

```
    6 CALL LGROOT(XT,NN,ALF,DPN,PN1,B,C,EPS)
      X(I) = XT
      A(I) = CC/DPN/PN1
C**   PUNCH 20, ALF,NN,I,XT,A(I)
      CSX = CSX + XT
    7 CSA = CSA + A(I)
C**   PUNCH 20, ALF,NN,I,CSX,CSA,TSX,TSA
   20 FORMAT(F6.2,2I3,2(1X,E14.8),2X,2(1X,E14.8))
      RETURN
      END
      SUBROUTINE LGROOT(X,NN,ALF,DPN,PN1,B,C,EPS)
      DIMENSION B(50),C(50)
      ITER = 0
    1 ITER = ITER +1
      CALL LGRECR(P,DP,PN1,X,NN,ALF,B,C)
      D = P/DP
      X = X-D
      IF(ABS(D/X)-EPS) 3,3,2
    2 IF(ITER-10) 1,3,3
    3 DPN=DP
      RETURN
      END
      SUBROUTINE LGRECR(PN,DPN,PN1,X,NN,ALF,B,C)
      DIMENSION B(50),C(50)
      P1=1.
      P = X-ALF-1.
      DP1=0.
      DP = 1.
      DO 1 J=2,NN
      Q = (X-B(J))*P - C(J)*P1
      DQ = (X-B(J))*DP + P - C(J)*DP1
      P1 = P
      P = Q
      DP1=DP
    1 DP = DQ
      PN = P
      DPN = DP
      PN1=P1
      RETURN
      END
      FUNCTION GAMMA(X)
      GAM(Y) = (((((((.035868343*Y-.193527818)*Y+.482199394)*Y-
     1 .756704078)*Y+.918206857)*Y-.897056937)*Y+.968205891)*Y
     2 -.577191652)*Y + 1.0
      Z = X
      IF(Z) 1,1,4
    1 GAMMA = 0.0
C**   PUNCH 2,Z
    2 FORMAT(2X,19HARG ERROR FOR GAMMA , E16.8)
      WRITE(6,2) Z
      GO TO 14
    4 IF(Z-70.) 6,1,1
    6 IF(Z-1.) 8,7,9
    7 GAMMA = 1.0
      GO TO 14
    8 GAMMA = GAM(Z)/Z
      GO TO 14
    9 ZA=1.0
   10 Z = Z-1.
      IF(Z-1.) 13,11,12
   11 GAMMA = ZA
      GO TO 14
```

```
12 ZA = ZA*Z
   GO TO 10
13 GAMMA = ZA*GAM(Z)
14 RETURN
   END
   SUBROUTINE LEG2HD(X,N,Q,Q1,IC)
C  A FORTRAN SUBROUTINE TO EVALUATE LEGENDRE FUNCTIONS OF THE SECOND
C  KIND AND THEIR DERIVATIVES
C  X=ARGUEMENT
C  N-3/2=MAXIMUN ORDER REQUIRED  (N LESS THEN OR EQUAL TO 105)
C  Q=LEGENDRE FUNCTION OF SECOND KIND
C  Q1=DERIVATIVE OF Q
C  IC=1 FOR JUST Q
C  IC=2 FOR Q AND Q1
   DIMENSION Q(125),Q1(125),R(125)
11 M=N+19
 6 FM=M
   IF(X-1.0014) 7,8,8
 7 Q(M+1)=RLEG2(X,M+1,19)
   Q(M)=RLEG2(X,M,19)
   GO TO 9
 8 Q(M)=1.
   Q(M+1)=(FM*X-SQRT(FM*FM*(X+1.)*(X-1.)+.25))/(FM+.5)
 9 DO 1 IJ=2,M
   I=M-IJ+2
   FI=I-1
   F=1.0/(FI-0.5)
   Q(I-1)=2.*FI*X*F*Q(I)-(F1+0.5)*F*Q(I+1)
   IF(Q(I-1)-1.E+30)1,1,10
10 N=N-10
   GO TO 11
 1 CONTINUE
   S=ALEG1(X,1)
   S=S/Q(1)
   DO 2 I=1,N
 2 Q(I)=S*Q(I)
   IF(IC-1) 4,4,3
 3 S=1./((X+1.)*(X-1.))
   Q1(1)=-.5*S*(X*Q(1)-Q(2))
   DO 5 I=2,N
   FI=I-1
 5 Q1(I)=(FI-.5)*S*(X*Q(I)-Q(I-1))
 4 RETURN
   END
   FUNCTION RLEG2(Z,N,M)
   QARG(U,FN,W,W2)=((W/(W+U))**FN-1.)/SQRT(U*(U+W2))
   W=Z+SQRT((Z+1.)*(Z-1.))
   W2=2.*(W-Z)
   FN=N
   FN=FN-.5
   RLEG2=0.
   U=.001
   DO 1 I=1,19
   RLEG2=RLEG2+QARG(U,FN,W,W2)
 1 U=U+.001
   A=QARG(.02,FN,W,W2)
   RLEG2=(.5*A+RLEG2)*.001+A*.005
   U=.03
   DO 2 I=3,M
   RLEG2=RLEG2+.01*QARG(U,FN,W,W2)
 2 U=U+.01
   RLEG2=RLEG2+.005*QARG(U,FN,W,W2)
```

```
      A=SQRT((Z-1.)*(Z+1.))
      B=-ALOG(A)
      C=ALOG(U+A+SQRT(U*(U+2.*A)))
      D=RLEG2+B+C
      RLEG2=D*W**(-FN)
      RETURN
      END
      FUNCTION ALEG1(X,N)
      DIMENSION Z(20),A(10),B(10)
      PI=3.1415926
      IF(X-1.09) 3,4,4
    4 DO 1 I=1,20
      FI=I*2-1
    1 Z(I)=COS(FI*PI*.025)
      ALEG1=0.0
      FN=N
      DO 2 I=1,20
      A1=1.0-Z(I)*Z(I)
      A2=X-Z(I)
    2 ALEG1=ALEG1+(A1/A2)**(N-1)/SQRT(A2)
      ALEG1=ALEG1/(2.0**(FN-0.5))*0.05*PI
      RETURN
    3 Y=X-1.0
      AL=ALOG(Y)
      A(1)=2.5*ALOG(2.)
      IF(N-1) 5,5,6
    6 NZ=N-1
      DO 7 I=1,NZ
      FI=I
    7 A(1)=A(1)-2./(2.0*FI-1.)
    5 B(1)=-.5
      FN=N-1
      G=FN*FN-.25
      DO 9 I=2,10
      FI=I-2
      H=I-1
      H2=.5/(H*H)
      H3=FI*H
      A(I)=(A(I-1)*(G-H3)-B(I-1)*(2.*G/H+1.))*H2
    9 B(I)=B(I-1)*(G-H3)*H2
      ALEG1=A(10)+AL*B(10)
      DO 10 I=1,9
      J=10-I
   10 ALEG1=A(J)+AL*B(J)+ALEG1*Y
      RETURN
      END


NUMBER OF CASES =     1
```

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